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Mergelyan's Approximation Theorem for Rational Modules*

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let $\bar{\partial} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$ be the Cauchy–Riemann operator in the complex plane $\mathbb{C}$. The main results of this paper are the following:

THEOREM 1. *Let X be a compact subset of $\mathbb{C}$ such that $\mathbb{C} - X$ has finitely many connected components. Suppose that g is a three-times continuously differentiable complex function in a neighbourhood of X, which satisfies $\bar{\partial} g(z) \neq 0$ for any $z \in X$. Then each $f \in C(X)$, satisfying $\bar{\partial}(\bar{\partial} f/\bar{\partial} g) = 0$ in $\mathring{X}$ in the weak sense can be uniformly approximated on X by functions ψ satisfying $\bar{\partial}(\bar{\partial}\psi/\bar{\partial} g) = 0$ in a neighbourhood of X.*

THEOREM 2. *Let X be a compact subset of $\mathbb{C}$ with finitely many complementary components, and let n be a nonnegative integer. Then each $f \in C(X)$ satisfying $\bar{\partial}^{n+1} f = 0$ in $\mathring{X}$ can be approximated by functions ψ satisfying $\bar{\partial}^{n+1}\psi = 0$ in a neighbourhood of X.*

The classical Mergelyan approximation theorem [6] is Theorem 2 with $n = 0$. Therefore the above results should be viewed as Mergelyan approximation theorems for the elliptic operators $\bar{\partial}(\bar{\partial}/\bar{\partial} g)$ and $\bar{\partial}^{n+1}$.

It is worth mentioning that results of this type are known [1, 2] to be true for a certain class of elliptic operators, whenever the compact X satisfies some strong cone conditions.

Theorems 1 and 2 are also related to approximation problems by rational modules. To explain this we introduce some notation. For a compact $X \subset \mathbb{C}$ let $R_0(X)$ denote the algebra of all rational functions with poles off X and $R(X)$ its uniform closure in $C(X)$. Throughout the work g will be

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a smooth function in a neighbourhood of X. If n is a positive integer, then we write $R(X, g, n)$ for the uniform closure in $C(X)$ of the $R_0(X)$-module

$$R_0(X, g, n) = R_0(X) + R_0(X)\, g + \cdots + R_0(X)\, g^n. \qquad (*)$$

When $n = 1$ we let $R(X, g)$ stand for $R(X, g, 1)$. Note that a function ϕ satisfies $\bar\partial(\bar\partial\phi/\bar\partial g) = 0$ in a neighbourhood of X if and only if $\phi = h + gk$ with h and k holomorphic, hence we can now restate Theorems 1 and 2 in the following way: If $X \subset \mathbb{C}$ is compact and has a finite number of complementary components, then

$$R(X, g) = \{ f \in C(X) / \bar\partial(\bar\partial f/\bar\partial g) = 0 \text{ in } \mathring{X} \},$$

whenever $g \in C^3(X)$ and $\bar\partial g(z) \neq 0$, $z \in X$, and moreover

$$R(X, \bar z, n) = \{ f \in C(X) / \bar\partial^{n+1} f = 0 \text{ in } \mathring{X} \}, \qquad n \geqslant 1.$$

The modules $R(X, \bar z, n)$ were introduced by O'Farrell [7] in connection with problems of rational approximation in Lipschitz norms. Later, several authors (e.g., Trent, Wang, and Verdera) have gone into the subject. In particular, the following result has been proved recently [10, 11, 4]. If $\mathring{X} = \varnothing$, then $R(X, g) = C(X)$ if and only if $R(Z) = C(Z)$, where Z denotes the subset of X on which $\bar\partial g$ vanishes.

At this point it is natural to look for versions of the above result which hold for arbitrary compact sets. The existence of interior points makes the problem more difficult (as in rational approximation) and the set Z is another obstruction when Z intersects ∂X. The best results we have been able to prove for the case with nonempty interior is the following extension of Theorem 1.

THEOREM 3. *Let X be a compact subset of $\mathbb{C}$ such that $\mathbb{C} - \bar{\mathring{X}}$ has finitely many connected components. Suppose, furthermore, that $Z \subset \mathring{X}$. Then $R(X, g) = \{ f \in C(X) /$ there are functions $h, k \in H(\mathring{X})$ so that $f = h + gk$ in $\mathring{X} \}$.*

It is worth mentioning that the hypothesis on the complementary components of X in Theorems 1 and 2 can be relaxed, with the same proof, to the following capacitary condition (γ is Ahlfors capacity [5]),

$$\gamma(D(z, r) - X) \geqslant cr,$$

for some positive constant c, for every $z \in \partial X$, and for all sufficiently small r. This condition is satisfied if the diameters of the complementary components of X are bounded away from zero. A similar remark applies to Theorem 3.

In Section 2, we prove Theorems 1 and 2. Section 3 contains the proof of

Theorem 3 and some other complementary results dealing with the case in which Z is nonempty.

We establish now some additional general notation. We denote by m the Lebesgue measure on $\mathbb{C}$, and by $C^i(U)$ (resp. $D^i(U)$), $i=1,2,\ldots$, the set of complex functions (resp. compactly supported functions) i-times continuously differentiable on the open subset U of $\mathbb{C}$. If U is an open subset of $\mathbb{C}$, then $H(U,g,n)$ will be the module $(*)$ replacing $R_0(X)$ by the space $H(U)$ of holomorphic functions in U. The symbol c stands for a positive constant, independent of the relevant variables under consideration, and not necessarily the same at each occurrence.

2. Proof of the Theorems

First we shall state a lemma and a consequence of it which we will need later. Their proofs can be found in [4].

LEMMA 1. *Let G be an open bounded subset of $\mathbb{C}$ with piecewise smooth boundary. Let U be an open set containing $\bar{G}$, and let $f, g \in C^2(U)$. Suppose that there exists a function $h \in C^1(\bar{G})$ such that $\bar{\partial} f = h \cdot \bar{\partial} g$. Then, for each $w \in G$, one has*

$$f(w)=\frac{1}{2\pi i}\int_{\partial G}\frac{f(z)}{z-w}\,dz-\frac{1}{2\pi i}\int_{\partial G}h(z)\,\frac{g(z)-g(w)}{z-w}\,dz$$
$$+\frac{1}{\pi}\int_G \bar{\partial} h(z)\,\frac{g(z)-g(w)}{z-w}\,dm(z).$$

If we suppose that $f\in D^2(\mathbb{C})$ satisfies the same hypothesis about $\bar{\partial} f$ as in Lemma 1 and that μ is a measure on $\mathbb{C}$ with compact support, then

$$\int f\,d\mu=\frac{1}{\pi}\int\bar{\partial}\left(\frac{\bar{\partial} f}{\bar{\partial} g}\right)\check{\mu}\,dm, \tag{1}$$

where we write

$$\check{\mu}(w)=\int\frac{g(z)-g(w)}{z-w}\,d\mu(z)\qquad\text{(see [4, 11]).}$$

Let us now recall some basic properties of Vituskin's localization operator (see [5, p. 210] for more information). If $f\in C(\mathbb{C})$ and $\psi\in D^1(\mathbb{C})$, we define

$$T_\psi(f)(z)=\frac{1}{\pi}\int\frac{f(w)-f(z)}{w-z}\,\bar{\partial}\psi(w)\,dm(w),\qquad z\in\mathbb{C}$$

The following properties hold:

(a) $T_\psi(f)$ is continuous on $\mathbb{C}$, $T_\psi(f)(\infty)=0$.

(b) $\bar{\partial} T_\psi(f)=\bar{\partial} f\cdot\psi$ in the weak sense.

(c) If $\operatorname{diam}(\operatorname{supp}\psi)\leqslant\delta$, then

$$\|T_\psi(f)\|_\infty\leqslant 2\delta\|\bar{\partial}\psi\|_\infty\,\omega(f,\delta),$$

where $\omega(f,\cdot)$ is the modulus of continuity of f.

The following lemma is analogous to a result due to Mergelyan (see [6; 9, p. 420]), essential in the original proof of his theorem on uniform approximation by rational functions.

LEMMA 2. *Let D be an open disc of radius $r>0$. Let E be a compact, connected subset of D with diameter at least r and such that $\Omega=(\mathbb{C}-E)\cup\{\infty\}$ is connected. Let g be a function of $D^1(\mathbb{C})$. Then there is a constant $c>0$ such that, for every $u\in\mathbb{C}$, there exists $Q(u,\cdot)\in H(\Omega,g)$ satisfying*:

(A) $|Q(u,z)|\leqslant c$,

(B) $|Q(u,z)-(g(z)-g(u))/(z-u)|\leqslant c\,r^3/|z-u|^3$,

for $z\in\Omega$, $u\in D$, and $z\neq u$.

Proof. Let $D=D(z_0,r)$. Let f_1 be a conformal representation of Ω in $D(0,1)$ such that $f_1(\infty)=0$. Since E is compact and connected, we have $\gamma(E)\leqslant\operatorname{diam}(E)\leqslant 4\gamma(E)$ [5, p. 199], where γ is the analytic capacity. Consequently, writing $a=f_1'(\infty)$, we get $|a|=\gamma(E)\geqslant r/4$. The function $f=f_1/a$: $\Omega\to D(0,1/|a|)$ is holomorphic and verifies

$$f(\infty)=0,\qquad f'(\infty)=1,\qquad \|f\|_\infty\leqslant 4/r. \tag{2}$$

Let Γ be the circle centred at z_0 and of radius r. We define

$$b=\frac{1}{2\pi i}\int_\Gamma (z-z_0)f(z)\,dz,\ b'=\frac{1}{2\pi i}\int_\Gamma (z-z_0)^2 f(z)\,dz.$$

Let us fix $u\in D$. We develop f, f^2, f^3 in a neighbourhood of ∞,

$$\begin{aligned} f(z)&=\frac{1}{z-u}+\frac{\lambda_2(u)}{(z-u)^2}+\frac{\lambda_3(u)}{(z-u)^3}+\cdots,\\ f^2(z)&=\frac{1}{(z-u)^2}+\frac{2\lambda_2(u)}{(z-u)^3}+\cdots,\\ f^3(z)&=\frac{1}{(z-u)^3}+\frac{3\lambda_2(u)}{(z-u)^4}+\cdots, \end{aligned} \tag{3}$$

if $|z-u|>2r$. Clearly,

$$\lambda_2(u)=\frac{1}{2\pi i}\int_{\Gamma_0}(z-u)f(z)\,dz,\ \lambda_3(u)=\frac{1}{2\pi i}\int_{\Gamma_0}(z-u)^2f(z)\,dz,$$

where Γ_0 is a circle centered at z_0, positively oriented and having a big enough radius.

Remembering that $1=f'(\infty)=(1/2\pi i)\int_{\Gamma_0}f(z)\,dz$, we obtain

$$\lambda_2(u)=b-u-z_0,\quad \lambda_3(u)=b'-2(u-z_0)\,b+(u-z_0)^2.$$

Let us define the following function, for $u\in\mathbb{C}$, $z\in\Omega$,

$$H(u,z)=f(z)+(u-z_0-b)f^2(z)$$
$$-(b'-2(u-z_0)\,b-2b^2-(u-z_0)^2)f^3(z).$$

The desired function is $Q(u,z)=(g(z)-g(u))\,H(u,z)$. Evidently $H(u,\ \cdot)\in H(\Omega)$ and so $Q(u,\cdot\,)\in H(\Omega,g)$.

Now we prove the estimates (A) and (B). By (2) and the definitions of b and b',

$$|b|\leqslant 4r,\qquad |b'|\leqslant 4r^2.$$

This and (2) give, for $z\in\Omega$ and $u\in D$,

$$|H(u,z)|\leqslant\frac{4}{r}+(|u-z_0|+|b|)\frac{16}{r^2}$$
$$+(|b'|+2|u-z_0|\,|b|+2|b|^2+|u-z_0|^2)\frac{64}{r^3}$$
$$\leqslant\frac{c}{r}. \tag{4}$$

For fixed $u\in D$, $H(u,z)(z-u)$ is holomorphic in Ω and the above estimate shows that it is bounded by c (independently of u) if $|z-u|\leqslant 2r$. By the maximum modulus principle, we conclude that $|H(u,z)|\,|z-u|\leqslant c$, $u\in D$, $z\in\Omega$. This and the mean-value theorem give (A). Taking into account (3), a calculation shows that

$$\left|H(u,z)-\frac{1}{z-u}\right|=\frac{1}{|z-u|^4}\,|h(u,z)|,\qquad z\in\Omega,u\in D,$$

where $h(u,\cdot)$ is a holomorphic function in $|z-u|>2r$ (including the infinity). If $z\in\Omega$ and $|z-u|\leqslant 2r$, then by (4) we get

$$|h(u,z)|=\left|H(u,z)-\frac{1}{z-u}\right|\,|z-u|^4\leqslant cr^3.$$

By the maximum modulus principle, the last inequality is valid for any $z \in \Omega$. Therefore

$$\left| Q(u,z) - \frac{g(z)-g(u)}{z-u} \right| \leqslant c\,|z-u| \left| H(u,z) - \frac{1}{z-u} \right|$$
$$\leqslant c \frac{r^3}{|z-u|^3}, \qquad z \neq u,\ z \in \Omega,\ u \in D,$$

which proves (B).

Let us note that he above lemma improves the conclusions of Mergelyan's lemma, losing, on the other hand, the holomorphy of $Q(u, \cdot)$. Also note that in Mergelyan's original lemma just the first and second coefficients of the development of f were considered.

LEMMA 3. *Let X be a compact of $\mathbb{C}$ and suppose that $\bar\partial g(z) \neq 0$ for every $z \in X$. Let $f \in C(X)$ such that $\bar\partial(\bar\partial f/\bar\partial g) = 0$ in $\mathring{X}$. Then for any $a \in \mathring{X}$*

$$\left| \frac{\bar\partial f}{\bar\partial g}(a) \right| \leqslant c \frac{\omega(f, d(a, \mathbb{C} - X))}{d(a, \mathbb{C} - X)}.$$

Proof. As already noted in the Introduction, we can write $f = h + gk$ with $h, k \in H(\mathring{X})$, so that $k = \bar\partial f/\bar\partial g$. Let us take $a \in \mathring{X}$. Let $r > 0$ such that $2r < d(a, \mathbb{C} - X)$. Let us choose a real, C^∞, even function ρ_1, with $0 \leqslant \rho_1 \leqslant 1$ and $\operatorname{supp} \rho_1 \subset [-4, 4]$ and such that $\rho_1 = 1$ on $[-1, 1]$. We put $\rho(z) = \rho_1(|z-a|^2/r^2)$ and $\psi = \rho/\bar\partial g$. Then ψ is of class C^2 and is supported on $D(a, 2r)$. If we consider the Vituskin's operator relative to f and ψ, then

$$\|T_\psi(f)\|_\infty \leqslant c\omega(f, 2r)\, r \|\bar\partial \psi\|_\infty.$$

On the other hand,

$$\|\bar\partial\psi\|_\infty = \left\| \bar\partial \left(\frac{1}{\bar\partial g} \right) \rho + \bar\partial \rho \frac{1}{\bar\partial g} \right\|_\infty \leqslant \frac{c}{r}.$$

Therefore

$$\left| \int_{C(a,r)} T_\psi(f)(z)\, dz \right| \leqslant cr\omega(f, 2r). \tag{5}$$

Applying Stokes' theorem and using property (c) of the operator T_ψ, we obtain

$$\int_{C(a,r)} T_\psi(f)\, dz = \int_{D(a,r)} \bar\partial f \psi\, d\bar z \wedge dz$$
$$= \int_{D(a,r)} k\, d\bar z \wedge dz = cr^2 k(a).$$

From (5) we conclude

$$|k(a)| \leqslant c\,\frac{\omega(f, 2r)}{r}.$$

Taking limits in the previous inequality when $r \to 1/2d(a, \mathbb{C}-X)$ the proof is completed.

Let us note that the above lemma implies the following: If $h_n + gk_n \in R_0(X, g)$, $n \in N$, is a sequence converging uniformly towards f on X, then $f = h + gk$ on $\mathring{X}$, h and k being analytic in $\mathring{X}$ and (h_n) (resp. (k_n)) converging uniformly towards h (resp. k) on compacts of $\mathring{X}$.

Proof of Theorem 1. Let $f \in C(X)$ with $\bar\partial(\bar\partial f/\bar\partial g) = 0$ in $\mathring{X}$, and let h, k as in Lemma 3, $f = h + gk$. First, we can assume that f and g are extend to $\mathbb{C}$ in such a way that supp f is included in the open set U in which $\bar\partial g \neq 0$. Let ρ be a function of class C^∞ such that supp $\rho \subset D(0, 1)$, $0 \leqslant \rho \leqslant 1$ and $\int \rho\, dm = 1$. If we write $\rho_\varepsilon(x) = (1/\varepsilon^2)\,\rho(x/\varepsilon)$, then (ρ_ε) is an approximate unit and

$$\int \rho_\varepsilon\, dm = 1, \qquad \int \bar\partial^i \rho_\varepsilon\, dm = 0, \qquad \text{supp}\, \rho_\varepsilon \subset \bar{D}(0, \varepsilon),$$

$$\int |\bar\partial^i \rho_\varepsilon|\, dm \leqslant \frac{c}{\varepsilon^2}, \qquad i = 1, 2. \tag{6}$$

Let μ be a measure on X which is orthogonal to $R(X, g)$. We write $f_\varepsilon(x) = f * \rho_\varepsilon(x) = \int f(x-t)\,\rho_\varepsilon(t)\, dm(t)$, $f_\varepsilon \in C^\infty(\mathbb{C})$. In the following we write F_ε instead $\bar\partial(\bar\partial f_\varepsilon/\bar\partial g)$. For a sufficiently small ε, supp f_ε is contained in U, so we can apply the integral formula (1); note that $\check\mu = 0$ because μ is orthogonal to $R(X, g)$. Thus we obtain

$$\int f_\varepsilon\, d\mu = \frac{1}{\pi}\int_{\mathbb{C}} F_\varepsilon \check\mu\, dm = \frac{1}{\pi}\int_X F_\varepsilon \check\mu\, dm. \tag{7}$$

So we must prove that the integral in (7) tends to zero when $\varepsilon \to 0$. Let us split this integral in two parts,

$$A_\varepsilon = \int_{X^\varepsilon} F_\varepsilon \check\mu\, dm, \qquad B_\varepsilon = \int_{X_\varepsilon} F_\varepsilon \check\mu\, dm,$$

where $X^\varepsilon = \{x \in X / d(x, \mathbb{C} - X) \geqslant 2\varepsilon\}$, $X_\varepsilon = \{x \in X / d(x, \mathbb{C} - X) < 2\varepsilon\}$. First, we shall estimate A_ε. If $x \in X^\varepsilon$, then $x - t \in \mathring{X}$ and $d(x-t, \mathbb{C} - X) \geqslant \varepsilon$ for $|t| < \varepsilon$. Besides

$$F_\varepsilon(x) = \int \bar\partial\left(\frac{\bar\partial f(x-t)}{\bar\partial g(x)}\right) \rho_\varepsilon(t)\, dm(t), \tag{8}$$

but, since $f(x-t)=h(x-t)+g(x-t)\,k(x-t)$, (8) turns into

$$\int \bar{\partial}\left(\frac{\bar{\partial}g(x-t)}{\bar{\partial}g(x)}\right) k(x-t)\,\rho_\varepsilon(t)\,dm(t).$$

By Lemma 3, the regularity of g and (6), we have

$$|F_\varepsilon(x)| \leqslant \frac{c}{\varepsilon} \sup_{\substack{x\in X\\ |t|<\varepsilon}} \left\{ \left|\bar{\partial}\left(\frac{\bar{\partial}g(x-t)}{\bar{\partial}g(x)}\right)\right| \omega(f, d(x-t, \mathbb{C}-X)) \right\} \leqslant c\,\|f\|_\infty. \quad (9)$$

Furthermore, for every x, $\lim F_\varepsilon(x)=0$ since

$$|F_\varepsilon(x)| \leqslant \sup_{\substack{x\\ |t|<\varepsilon}} \left|\bar{\partial}\left(\frac{\bar{\partial}g(x-t)}{\bar{\partial}g(x)}\right)\right| c(x) \leqslant \varepsilon c(x).$$

By [4] $|\check{\mu}| \leqslant c$; then the dominated convergence theorem implies that $\lim A_\varepsilon = 0$.

Now we are going to estimate B_ε. If $x \in X_\varepsilon$, it can be deduced from (6) that

$$\bar{\partial}^i f_\varepsilon(x) = \int (f(x-t)-f(x))\,\bar{\partial}^i \rho_\varepsilon(t)\;dm(t), \qquad i=1,2,$$

so

$$|\bar{\partial}^i f_\varepsilon(x)| \leqslant c\frac{\omega(f,\varepsilon)}{\varepsilon^2}, \qquad i=1,2.$$

Consequently

$$|F_\varepsilon(x)| \leqslant c\,\frac{\omega(f,\varepsilon)}{\varepsilon^2},$$

and we conclude that

$$|B_\varepsilon| \leqslant c\,\frac{\omega(f,\varepsilon)}{\varepsilon^2} \int_{X_\varepsilon} |\check{\mu}|\;dm. \quad (10)$$

As $\mathbb{C}-X$ has a finite number of components, for a sufficiently small ε we can cover X_ε by means of discs $D_1,\ldots,D_n$ of radius 4ε whose centres do not lie en X. Evidently every disc contains a compact, connected subset $E_i \subset D_i$ such that $E_i \subset \mathbb{C}-X$ and $\operatorname{diam}(E_i) \geqslant \varepsilon$. Applying Lemma 2, we obtain functions $Q_j(u,\cdot) \in H(\mathbb{C}-E_j, g)$ (and so in $R(X,g)$), satisfying

$$|Q_j(u,z)| \leqslant c, \quad (11)$$

$$\left|Q_j(u,z) - \frac{g(z)-g(u)}{z-u}\right| \leqslant c\,\frac{\varepsilon^3}{|z-u|^3}, \quad (12)$$

for $z \in \mathbb{C}-E_j$ and $u \in D_j$.

If we define $L_1 = D_1 \cap X_\varepsilon$ and $L_j = D_j \cap X_\varepsilon - (L_1 \cup \cdots \cup L_{j-1})$ for $1 < j \leqslant n$, then $X_\varepsilon = L_1 \cup \cdots \cup L_n$ (disjoint union). For every $u \in X_\varepsilon$ there is a unique j such that $u \in L_j$ and then

$$|\check{\mu}(u)| \leqslant \int_X \left| Q_j(u, z) - \frac{g(z) - g(u)}{z - u} \right| d|\mu|(z).$$

The integral in (10) is bounded as follows

$$\sum_j \int_{L_j} |\check{\mu}(u)|\, dm(u) \leqslant \int_X d|\mu|(z) \sum_j \int_{L_j} \left| Q_j(u, z) - \frac{g(z) - g(u)}{z - u} \right| dm(u). \tag{13}$$

Fixing $z \in X$, we get

$$\begin{aligned} &\sum_j \int_{L_j} \left| Q_j(u, z) - \frac{g(z) - g(u)}{z - u} \right| dm(u) \\ &\quad \leqslant \sum_j \int_{L_j \cap D(z, 4\varepsilon)} \left| Q_j(u, z) - \frac{g(z) - g(u)}{z - u} \right| dm(u) \\ &\quad\quad + \sum_j \int_{L_j \cap (\mathbb{C} - D(z, 4\varepsilon))} \left| Q_j(u, z) - \frac{g(z) - g(u)}{z - u} \right| dm(u) \leqslant (*). \end{aligned}$$

Let us estimate the first integral by (11) and the second by (12),

$$\begin{aligned} (*) &\leqslant \sum_j \int_{L_j \cap D(z, 4\varepsilon)} c\, dm(u) + \sum_j \int_{L_j \cap (\mathbb{C} - D(z, 4\varepsilon))} c \frac{\varepsilon^3}{|z - u|^3}\, dm(u) \\ &\leqslant c\varepsilon^2 + c \int_{4\varepsilon}^{\infty} \int_0^{2\pi} \rho \frac{\varepsilon^3}{\rho^3}\, d\rho\, d\theta = c\varepsilon^2. \end{aligned}$$

Combining the previous estimation with (10) and (13), we obtain

$$|B_\varepsilon| \leqslant c \frac{\omega(f, \varepsilon)}{\varepsilon^2} |\mu|(X)\, \varepsilon^2 = O(\omega(f, \varepsilon)).$$

Therefore $\lim A_\varepsilon + B_\varepsilon = 0$ and so $\int f\, d\mu = 0$. This completes the proof.

Remark (J. Verdera). The regularity of g in Theorem 1 ($g \in C^3$) has only been used in the estimate (9) to prove that $\lim A_\varepsilon = 0$. What happens if we only require that $g \in C^{2+\alpha}$ with $0 \leqslant \alpha \leqslant 1$? Using that k is holomorphic, we can improve (9) as follows:

$$\sup_{\substack{x \in X^\varepsilon \\ |t| \leqslant \varepsilon}} |k(x - t)| = \sup_{\varepsilon \leqslant d(y, \mathbb{C} - X) \leqslant 2\varepsilon} |k(y)| \leqslant c \frac{\omega(f, 2\varepsilon)}{\varepsilon},$$

so if $g \in C^{2+\alpha}$, the functions $f \in \mathrm{Lip}(1-\alpha, X)$ can be approximated, because

$$|F_\varepsilon(x)| \leqslant c \sup_{\substack{x \in X \\ |t| \leqslant \varepsilon}} \left| \bar{\partial} \left(\frac{\bar{\partial} g(x-t)}{\bar{\partial} g(x)} \right) \right| \frac{\omega(f, 2\varepsilon)}{\varepsilon} \leqslant c\varepsilon^\alpha \frac{\varepsilon^{1-\alpha}}{\varepsilon} = O(1),$$

and $|F_\varepsilon(x)| \leqslant c(x)\, \varepsilon^\alpha$.

If $\alpha = 0$, then $\sup_{x, |t| \leqslant \varepsilon} |\bar{\partial}(\bar{\partial} g(x-t)/\bar{\partial} g(x))|$ tends to zero by the uniform continuity of $\bar{\partial}^2 g$ on X.

The proof of Theorem 2 is essentially analogous to the previous one, but the details are more involved. We will just state the required lemmas whose proofs can be found in [3].

Lemma 4. *If $f \in D(\mathbb{C})$ and $n \geqslant 1$, then*

$$f(w) = \frac{(-1)^{n+1}}{n!\, \pi} \int \bar{\partial}^{n+1} f(z) \frac{(\bar{z} - \bar{w})^n}{z - w}\, dm(z), \qquad w \in \mathbb{C}.$$

Besides, if μ is a measure with compact support on $\mathbb{C}$, then

$$\int f\, d\mu = \frac{(-1)^{n+1}}{n!\, \pi} \int \bar{\partial}^{n+1} f \check{\mu}_n\, dm,$$

where

$$\check{\mu}_n(w) = \int \frac{(\bar{z} - \bar{w})^n}{z - w}\, d\mu(z).$$

The analogue of Lemma 2 for the operator $\bar{\partial}^{n+1}$ is

Lemma 5. *Let D be an open disc of radious $r > 0$, E a subset of D compact, connected and of diameter al least r, such that $\Omega = \mathbb{C} - E$ is connected. Let $n \geqslant 1$. For every $u \in \mathbb{C}$ there is a function $T(u, \cdot) \in H(\Omega, \bar{z}, n)$ and a constant c (which depends only on n) such that*

(A) $|T(u, z)| \leqslant c |z - u|^{n-1}$,

(B) $|T(u, z) - (\bar{z} - \bar{u})^n/(z - u)| \leqslant c r^{n+2}/|z - u|^3$, $z \neq u$, $\quad z \in \Omega$, *and* $u \in D$.

Let us point out that for $n = 0$, we obtain Mergelyan's classical lemma.

It is worthwhile observing that in the proof of Theorem 2, the part corresponding to A_ε is zero (since $\bar{\partial}^{n+1}$ commutes with the convolution), therefore in this case it is only necessary to bound B_ε.

3. The Case $Z \neq \emptyset$

In this section we will prove an extension of Theorem 1 in the case that $Z = \{x \in X / \bar{\partial} g(x) = 0\}$ is not empty. We begin with a lemma giving necessary conditions for $f \in R(X, g)$. Throughout this section, except in Theorem 3, g is only assumed to be of class C^2.

Lemma 6. *Let $f \in R(X, g)$. Then*

(i) *There are functions h and k holomorphic in $\mathring{X} - Z$ such that $f = h + gk$ in $\mathring{X} - Z$.*

(ii) *If $Z \subset \mathring{X}$, then the functions h and k are holomorphic in $\mathring{X}$.*

(iii) *$f_{|Z} \in R(Z)$.*

Proof. (i) is clear and was already used before. To prove (ii), we consider a finite union $L \subset \mathring{X}$ of piecewise smooth curves such that for every $z \in Z$ $\operatorname{Ind}_L(z) = 1$. Let ρ be a function of class C^∞ vanishing in a neighbourhood of Z and equal to 1 in a neighbourhood of L. Put $\psi = \rho / \bar{\partial} g$. For a sufficiently small r, the function ρ will be equal to 1 in $D(a, r)$, for any $a \in L$. Let $f \in R(X, g)$ and $f_n = h_n + g k_n \in R_0(X, g)$ such that $f_n \to f$ uniformly on X. As in Lemma 3, we obtain

$$\int_{C(a,r)} T_\psi(f_n)\, dz = c r^2 k_n(a), \qquad a \in L. \tag{14}$$

Therefore by property (c) of the operator T_ψ (Section 2) and (14) we have

$$|k_n(a)| \leqslant c(\psi, r) \| f_n \| = O(1), \qquad a \in L. \tag{15}$$

By the maximum modulus principle, the inequality (15) is true in a neighbourhood W of Z. Then there is a subsequence, also denoted (k_n), converging towards an holomorphic function $\tilde{k}$ in W. By (i) $f = h + gk$ in $\mathring{X} - Z$. Since $f_n \to f$ uniformly on X and $\bar{\partial} f_n = \bar{\partial} g \cdot k_n \to \bar{\partial} g \cdot \tilde{k}$ uniformly in W, we conclude that $\bar{\partial} f = \bar{\partial} g \cdot \tilde{k}$ in the weak sense in W. So $k = \tilde{k}$ in $W - Z$. Defining $\tilde{h} = f - g\tilde{k}$, we deduce that both h and k have holomorphic extensions to Z. To prove (iii) it is enough to observe that $g_{|Z} \in R(Z)$, whenever $g \in C^1$ and $\bar{\partial} g = 0$ in Z [5, p. 26].

The following proposition is a partial converse of the above lemma.

Proposition. *Suppose that f is a function of class C^2 in a neighbourhood of X and*

(a) *there are holomorphic functions h and k such that $f = h + gk$ in $\mathring{X}$;*

(b) *f is holomorphic in a neighbourhood of $Z \cap \partial X$. Then $f \in R(X, g)$.*

Proof. Let U be the open set in which f is defined and $U_1 \subset U$ the open in which f is analytic. Let $S = \{w \in U / \bar{\partial} g(w) = 0\}$. Let us choose $\varepsilon > 0$ smaller than the distances between $(S - U_1) \cap (\mathbb{C} - X)$ and X, between X and $\mathbb{C} - U$ and between $Z \cap \partial X$ and U_1. Let us consider points $x_1, ..., x_n$ in X such that

$$G = D(x_1, \varepsilon) \cup \cdots \cup D(x_n, \varepsilon) \supset X.$$

Then G is a bounded open set with piecewise smooth boundary and it is defined so that the function $\bar{\partial} f / \bar{\partial} g$ is of class C_1 in $\bar{G}$. We can apply Lemma 1 to obtain the equality

$$f(w) = \frac{1}{2\pi i} \int_{\partial G} \frac{f(z)}{z - w} \, dz - \frac{1}{2\pi i} \int_{\partial G} \frac{\bar{\partial} f}{\bar{\partial} g}(z) \frac{g(z) - g(w)}{z - w} \, dz$$

$$+ \frac{1}{\pi} \int_G \bar{\partial} \left(\frac{\bar{\partial} f}{\bar{\partial} g} \right)(z) \frac{g(z) - g(w)}{z - w} \, dm(z), \qquad w \in X. \tag{16}$$

Let μ be an orthogonal measure to $R(X, g)$. Integrating in (16) and applying Fubini's theorem, we obtain

$$\int f \, d\mu = \frac{1}{2\pi i} \int_{\partial G} f \hat{\mu} \, dz - \frac{1}{2\pi i} \int_{\partial G} \frac{\bar{\partial} f}{\bar{\partial} g} \check{\mu} \, dz$$

$$+ \frac{1}{\pi} \int_G \bar{\partial} \left(\frac{\bar{\partial} f}{\bar{\partial} g} \right) \check{\mu} \, dm.$$

where $\hat{\mu}(z) = \int d\mu(w)/(w - z)$ is the usual Cauchy transform of μ [5, p. 46].

The measure μ is orthogonal to $R(X, g)$, and so $\hat{\mu} = 0$ and $\check{\mu} = 0$ in $\mathbb{C} - X$; therefore

$$\int f \, d\mu = \frac{1}{\pi} \int_X \bar{\partial} \left(\frac{\bar{\partial} f}{\bar{\partial} g} \right) \check{\mu} \, dm.$$

From the fact that $\check{\mu}$ is continuous off a countable set [4, 11], we derive that $\check{\mu} = 0$ a.e. dm in $\mathbb{C} - \mathring{X}$. So the domain of integration in the last integral is reduced to $\mathring{X}$, in which $\bar{\partial}(\bar{\partial} f / \bar{\partial} g) = 0$. Consequently, $\int f \, d\mu = 0$ and so $f \in R(X, g)$.

COROLLARY 1. *Suppose that $Z = \varnothing$ (resp. $Z \cap \partial X = \varnothing$) and f a function of class C^2 in a neighbourhood of X. Then $f \in R(X, g)$ if and only if $\bar{\partial}(\bar{\partial} f / \bar{\partial} g) = 0$ in $\mathring{X}$ (resp. $f = h + gk$ with $h, k \in H(\mathring{X})$).*

Observe the analogy of Corollary 1 with the classical result stating that $f \in R(X)$ whenever $\bar{\partial} f = 0$ on X. Yet here it is assumed that the operator applied to the function vanishes in $\mathring{X}$ and not necessarily in X.

COROLLARY 2. *Suppose* $\mathring{X}=\varnothing$. *Then* $R(X,g)=C(X)$ *if and only if* $R(Z)=C(Z)$.

Proof. It is enough to note that if $R(Z)=C(Z)$ and $\mathring{X}=\varnothing$, then every function of $C^2(\mathbb{C})$ is uniform limit in X of functions of $C^2(\mathbb{C})$ which are analytic in a neighbourhood of Z.

The aim of the following lemma is to reduce the study of $R(X,g)$ to that of $R(\overline{\mathring{X}}\, g)$.

LEMMA 7. *A measure* μ *on* X *is orthogonal to* $R(X,g)$ *if and only if* μ *is supported in* $\overline{\mathring{X}}\cup Z$ *and* μ *is orthogonal to* $R(\overline{\mathring{X}}\cup Z, g)$.

The proof of this lemma, which we omit, is essentially based in the good continuity properties of $\check{\mu}$ (see [3]).

Now we point out that the modules $R(X,g)$, under certain restrictions, are local [5, p. 50], i.e., if $f\in C(X)$ and for every $x\in X$ there is a neighbourhood U_x such that $f\in R(\bar{U}_x, g)$, then $f\in R(X,g)$.

LEMMA 8. *The following statements hold*:

(i) *If* Z *is finite, then* $R(X,g)$ *is local.*

(ii) *Let* $U_1,\dots,U_n$ *be a cover of* X *such that* $Z\cap\partial U_i=\varnothing$ $(i=1,2,\dots,n)$. *If* $f\in C(X)$ *and* $f\in R(\bar{U}_i,g)$, *then* $f\in R(X,g)$.

The previous result was proved in the case $g(z)=\bar{z}$ by O'Farrell [8] and Weinstock [14]. If g is a polynomial in $\bar{z}$ the proof is in [11]. The general case follows. A different proof can be found in [3].

Proof of Theorem 3. The inclusion $\subset$ is a consequence of Lemma 6. First, we suppose that $X=\overline{\mathring{X}}$ and that $\mathbb{C}-X$ has finitely many components. Let W be an open set such that $Z\subset W\subset\bar{W}\subset\mathring{X}$ and $f\in C(X)$, $f=h+gk$ in $\mathring{X}$, with $h,k\in H(\mathring{X})$. If $x\notin Z$ and U_x is a neighbourhood of x such that $U_x\cap Z=\varnothing$, then $f\in R(\bar{U}_x\cap X, g)$ by Theorem 1. Cleary $f\in R(\bar{W},g)$ and so Lemma 8 gives that $f\in R(X,g)$. The case in which $\mathbb{C}-\overline{\mathring{X}}$ has a finit number of components can be reduced to the previous one by Lemma 7. The general case is obtained through a standard argument of localization [5, p. 51].

4. Open Questions

Now we pose some questions whose solutions are unknown to us.

(A) *Is there a compact* X *of* $\mathbb{C}$ *such that* $R(X,\bar{z})\neq\{f\in C(X)/\bar{\partial}^2 f=0$ *in* $\mathring{X}\}$?

We think that the answer is affirmative but we have not been able to find a counterexample. If (A) is true, it makes sense to wonder:

(B) *Which compacts* X *of* $\mathbb{C}$ *satisfy that* $R(X, \bar{z}) = \{ f \in C(X)/\bar{\partial}^2 f = 0$ *in* $\mathring{X}\}$?

It seems that an answer to (B) could be obtained using Vituskin's techniques [5] adapted to the operator $\bar{\partial}^2$.

(C) *Is* $R(X, g)$ *always local?*

There is some evidence in favour of localness (apart from Lemma 8), but the difficulty is to find a way of avoiding Z.

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